Mathematics of Choice

Solutions Manual

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1 Introductory Questions

1. A sample year, where the 13th of January is Sunday has a 13th of February of Wednesday and so on. We can represent each day of the week as an integer (1 = Sunday, 2 = Monday, ..., 7 = Saturday), and each month's 13th with the corresponding integer i.e. in this case January would be 1 because it has a 13th on Sunday, and Februrary would be 4 because it has a Wednesday on its 13th. We then get the following list of integers:

   $$1, 4, 4, 7, 2, 5, 7, 3, 6, 1, 4, 6$$

   Let us now adjust the parameters and take *Friday* to be 1. In this case, the first 13th of the month is Friday, the second is Monday and so is the third, and so on. In total, there are 2 Friday the 13ths this year. If we run this sample for each possible day (7 possible days), we find out that there can be a maximum of 3 Friday the 13ths in a year.

2. There are 6 faces to a cube. In this case, a cube is said to be different to another if they can *not* be rotated in such a way to resemble each other. We can consider the cube by color count to solve this: there is only **1** possible cube with 6 blue or red faces; a cube with 5 blues or reds and 1 red or blue can be arranged in only **1** way as well; a cube with 4 blues or reds and 2 reds or blues can be arranged in **2** ways, where the faces with the lesser color are adjacent *or* opposite to each other; a cube with 3 blues and 3 reds has can only be arranged in **1** distinct way. This gives us a sum of 10 possible combinations.

3. Consider that this man is at an origin point of coordinates (0,0). His destination is (7,8). Let us assume he can only move up, or right. He makes 15 decisions in total; of these 7 must be right and 8 up. Encoding his decisions as a string gives us: 'RRRRRRRUUUUUUUU'. We must decide where the R's go, or where the U's go, and the possible combinations thus are the total possible paths. The

result:

$$\binom{15}{8} = \binom{15}{7} = 6435 \text{ possible paths}$$

4. Label the twenty possible books in the format b_n. The governor makes a total of ten maximum decisions; in each decision, he can pick from b_1 to b_{20}.

   (a) He can pick the same book repeatedly, meaning *no replacement*. The answer is thus, according to the multiplication rule, 20^{10} possible non-distinct book combinations.

   (b) He can pick only ten distinct books, meaning *with replacement*. We can thus use the binomial coefficient:

   $$\binom{20}{10} = 184756 \text{ possible choices}$$

5. To be solved later.

2 Permutations and Combinations

2.1 The Multiplication Principle

1. The process here is simple: first, we pick our first letter, say 'B', from four choices. Then we pick our second, say 'A', from three choices. Now we pick from two choices, 'C' and 'D', our third letter; let's say 'C'. Finally, we have 'D' left and that falls into place. In so doing, we have constructed one possible arrangement: 'BACD'. Using the multiplication rule, we can express this as thus: $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possible choices.

2. From 26 choices of letters, we must pick the first letter; then, from the same 26, we must pick the second and the third as well. In terms of the multiplication principle: $26 \cdot 26 \cdot 26 = 17576$ possible license plates.

3. $26 \cdot 25 \cdot 24 = 15600$ possible license plates.

4. $1 \cdot 26 \cdot 26 = 676$ possible license plates.

5. $1 \cdot 25 \cdot 24 = 600$ possible license plates *beginning* with Q, and $26 \cdot 25 \cdot 1 = 650$ possible license plates *ending* with Q.

6. $26 \cdot 26 \cdot 5 = 3380$ possible license plates.

7. $5 \cdot 25 \cdot 24 = 3000$ ways.

8. $6 \cdot 5 = 30$ ways.

9. $8 \cdot 2 \cdot 2 \cdot 2 = 64$ possible tires.

10. $23 \cdot 12 \cdot 3 \cdot 6 = 4968$ kinds of slippers.

11. There are 5 digits in any number between 10,000 and 100,000. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$ if only 6, 7, or 8. $3 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 768$ if 6, 7, 8, and 0 (since 0 can not start the number without reducing it to 4 digits).

2.2 Factorials

1. (a) $3! = 3 \cdot 2 \cdot 1 = 6$
   (b) $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
   (c) $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$

2. (a) $\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \ldots}{10 \cdot 9 \cdot 8 \ldots} = 12 \cdot 11 = 132$
   (b) $2! = 2 \cdot 1 = 2$
   (c) $4! + 3! = 24 + 6 = 30$
   (d) $7! = 5040$

2.3 Permutations

1. (a) $P(7,3) = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$
   (b) $P(8,4) = \frac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$
   (c) $P(20,2) = \frac{20!}{18!} = 20 \cdot 19 = 380$

2. (a) i. $P(7,3) = P(15,2)$
      ii. $\frac{7!}{4!} = \frac{15!}{13!}$
      iii. $7 \cdot 6 \cdot 5 = 15 \cdot 14$
      iv. $210 = 210$
   (b) i. $P(6,3) = P(5,5)$
      ii. $6 \cdot 5 \cdot 4 = 5!$
      iii. $120 = 120$

3.

$$\begin{aligned}P(n,1)+P(m,1)&=P(n+m,1)\\ \frac{n!}{(n-1)!}+\frac{m!}{(m-1)!}&=\frac{(n+m)!}{(n+m-1)!}\\ \frac{n(n-1)!}{(n-1)!}+\frac{m(m-1)!}{(m-1)!}&=\frac{(n+m)(n+m-1)!}{(n+m-1)!}\\ n+m&=n+m\end{aligned}$$

4.

$$\begin{aligned}P(n,n)&=P(n,n-1)\\ n!&=\frac{n!}{(n-n+1)!}\\ n!&=\frac{n!}{1!}\\ n!&=n!\end{aligned}$$

5. $P(24,3)=\frac{24!}{21!}=12144$ different names

6. In the first case, where repetitions are allowed: $24\cdot 24\cdot 24=13824$ different names. In the second case, where repetitions are allowed and two-letter words must be counted as well: $(24\cdot 24\cdot 24)+(24\cdot 24)=13824+576=14400$ different words.

7. The first digit cannot be a 0 (that confused me for a while). Therefore:

$$\begin{aligned}9\cdot 9\cdot 8\cdot 7&=4536 \text{ numbers with distinct digits}\\ 5\cdot 8\cdot 8\cdot 7&=2240 \text{ odd numbers with distinct digits}\end{aligned}$$

8.

$$\begin{aligned}P(5,4)&=120 \text{ numbers with distinct digits}\\ 3\cdot 4\cdot 3\cdot 2&=72 \text{ odd numbers with distinct digits}\end{aligned}$$

9. (a) $6\cdot 6\cdot 5\cdot 4=720$ numbers with distinct digits.

   (b) Where the number ends in 0: $1\cdot 6\cdot 5\cdot 4=120$ and where it ends with a non-zero digit: $3\cdot 5\cdot 5\cdot 4=300$. For a total 420 even numbers with distinct digits.

10. If we have at our disposal only eight digits, numbers past that length in digits (i.e., nine-digit or ten-digit numbers) will have repeating digits (violating property two). Therefore, we can only consider numbers greater than 53,000 and less than or equal to eight digits.

    Let us go at these numbers by digit count. For eight digits, there are $P(8,8) =$ 40320 numbers. For seven digits, there are $P(8,7) = 40320$ numbers. For six digits, there are $P(8,6) = 20160$ numbers. For five digits, we must be more specific (as 53000 is a five-digit number). Specifically, there are two possibilities in this regard: 1. numbers between 60000 and 100000; 2. numbers between 53000 and 59999.

    For the first possibility, we have: $3 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 2520$. For the second: $1 \cdot 5 \cdot 6 \cdot 5 \cdot 4 = 600$ numbers. Summing all this up gives us: $130,920$ numbers.

2.4 Combinations

1. (a) $C(6,2) = \frac{6!}{2!4!} = \frac{6 \cdot 5 \cdot 4!}{2!4!} = \frac{30}{2} = 15$
   (b) $C(7,4) = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!3!} = \frac{210}{6} = 35$
   (c) $C(9,3) = \frac{9!}{3!6!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3!6!} = \frac{504}{6} = 84$

2. $C(6,2)$ and $C(6,4)$ follow the property $C(n,r) = C(n,n-r)$. We can show this thus (bracketed are the 4-subsets):

$$AB(CDEF),\ AC(BDEF),\ AD(BCEF),\ AE(BCDF),\ AF(BCDE)$$
$$BC(ADEF),\ BD(ACEF),\ BE(ACDF),\ BF(ACDE)$$
$$CD(ABEF),\ CE(ABDF),\ CF(ABDE)$$
$$DE(ABCF),\ DF(ABCE)$$
$$EF(ABCD)$$

3. (a) Label the ten questions '0'-'9' giving us the set: '0123456789'. The question is how many different *combinations* we can have of these, taken 8 at a time (because the order they are picked in is irrelevant). This is $C(10,8) = 45$.
   (b) This is the complement of part (a), $C(10,2)$ and thus has the same answer 45.

4. $C(720,10) = \frac{720!}{(710)!10!}$

5.

$$\begin{aligned} C(n,r) &= C(n,n-r) \\ \frac{n!}{(n-r)!r!} &= \frac{n!}{(n-(n-r))!(n-r)!} \\ \frac{n!}{(n-r)!r!} &= \frac{n!}{(n-r)!r!} \end{aligned}$$

6. (a) We must count each two collinear points, and only once since order is irrelevant, and those two points are collinear only to each other. So the answer would be $C(20,2) = 190$

   (b) $C(20,3) = 1140$

7. Instead of looking for a direct method, we can consider the complement of the question: in how many arrangements are the two people *together*? Let us call this c; subtracting c from the total number of arrangements 10! will give us our answer n: $10! - c = n$. By considering the two people as a unit, we get the following statement for $c = 2 \cdot 9!$. It is multiplied by two in consideration of the fact that the two people are distinct. Substituting this in gives us our answer: $n = 10! - 2 \cdot 9! = 2,903,040$ arrangements.

8. Let n be any positive integer. This theorem asserts that $n(n+1)(n+2)(n+3)(n+4)$ is divisible by 5!. We can express this as a binomial coefficient:

$$C(n+4,5) = \frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$$

   . $C(n+4,5)$ *must* be a whole number, since it is a binomial coefficient. This means that the result of $\frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$ is a whole number, and thus the theorem is true.

9. (a) 9! orders

   (b) Consider the four red books one unit, and the five green books one unit. This gives us 2! orders therein. But we also have to consider the internal orders of the books, 4! and 5!. The product of the three is our answer: $2!4!5! = 120$ orders.

   (c) Consider the red books as one unit (accounting for internal order), but not the green: $4!6! = 17,280$ orders.

   (d) 5!4! orders.

10. (a) From thirty businessmen, three; and from thirty professors, three; the last can be any two from the remaining in either group. $C(30,3) \cdot C(30,3) \cdot C(54,2)$ ways.
    (b) $C(30,1) \cdot C(59,7)$ ways.

11. For each position, the citizen is presented with a set number of options: they can choose not to vote; or they can pick a candidate. For the vote for mayor, this comes out to four choices (3 candidates + decision not to vote) and so on. This can be represented as $4 \cdot 5 \cdot 4 - 1 = 79$. We subtract one because that is the invalid arrangement where the voter chooses not to vote for all three positions.

12. There are four, because you can construct four 10s with the factors of 20!.

13. Twelve.

14. 7^5. There is an unlimited supply, but finite (7 colors). The only decision is of which color flag to run up; since they are unlimited, the number of colors does not decrease.

15. (a) Going at the flags from the bottom: $7 \cdot 6 \cdot 6 \cdot 6 \cdot 6$.
    (b) $P(7,5) = 2520$

16. ...

17. Distinct boxes and objects means order *matters*. This is simply permutation, then:

$$P(k,n) = \frac{k!}{(k-n)!}$$

2.5 Permutations of Things in a Circle

1. $(8-1)! = 5040$ ways.

2. $\frac{1}{8}(8! - 2 \cdot 7!) - \frac{1}{8}(2 \cdot 1 \cdot 6!) = 3600$ ways.

3. Let us fix one lady. That leaves three more ladies, and four men. They can be permuted in alternate order in $4!3! = 144$ ways.

4. ...

5. $5! = 120$ ways

6. $5! = 120$ ways

7. Consider the complementary pairs that sum to 7 (6 and 1, 5 and 2, 4 and 3) as one. That gives us $\frac{1}{6}(3!2!) = 2$ circular permutations.