Notes on 'How to Solve It, by George Polya'

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1 How to Solve It

1.1 The Heuristic

As you attempt a mathematical problem, you develop your (orginally incomplete) understanding of the question given. This is a continuous process, and is occurring even when you are close to solving said problem. It can be broadly grouped into **four** phases:

1. Understand the problem.
2. Determine the connection between the unknown and the data, and gleam from it a plan.
3. Carry out your plan.
4. Review your solution.

Step 1. The first step is **understanding the problem and figuring out what is required of *us***. To this end, you should be able to:

1. **Restate the problem fluently**, perhaps to the point that you can read it out of memory.
2. **Understand** the unknowns, the data, the condition (for proofs, understand the theorem), and the line linking them.
3. To begin, **is satisfying the condition even possible** (with the data we have)?
4. You should **draw out any figures**.
5. If not given, you should **introduce suitable notation**.

Step 2. Now that we understand *how* the various 'items' are connected, **we make a plan**. Planning is essentially a matter of experience — the more problems you've solved (especially those related to the question at hand), the easier this will come to you. Planning involves:

1. **Recalling similar problems** we have solved before, especially ones with similar unknowns.
2. If that doesn't click, you can **try restating the problem** — generalizing it, specializing it, analogizing it, removing parts of it, e.t.c.

Step 3. The final *working* step is to **carry out the plan**. This is simple, so long as one does not lose sight of the solution — something unlikely assuming you reasoned the solution himself. Take care and patience, to ensure you do not make any **silly mistakes** and **errors in calculation**.

Step 4. There is still one phase left: **reviewing our solution**. This is an oft neglected part of problem-solving. In reviewing:

1. We **double-check** for any errors we have made.
2. Understand the solution such that we may use it in solving **other problems** in phase two.
3. We ask ourselves: could we have **derived this solution with a different method**?
4. Also: is there a **more concise way** to explain our solution?

We will now apply this heuristic to an example problem.

1.2 Example

Example 1. A rectangular garden has a length that is 5 meters more than twice its width. If the area of the garden is 60 square meters, find the dimensions of the garden.

Step 1:

1. **What are the unknowns**? The dimensions of the garden, which we can notate with l (length) and w (width).
2. **What is the data**? That $l = 2w + 5$, and the area, A, of the garden is 60 m^2.
3. **What is the condition** ? That $l = 2w + 5$ and $l \cdot w = 60$, since this links all our data and unknowns together.

Step 2:

1. **Have you solved a similar problem before**? Yes, I have. Algebraically, this problem resembles a *simultaneous equation*, where we have two unknowns and two equations linking them.
2. **If not, try restating the problem**. If you knew the width, could you solve this? What if it were a square instead of a rectangle?

Step 3:

1. We have two equations: the first, $l = 2w + 5$; and the second, $l \cdot w = 60$.
2. One of our variables is already expressed in terms of the other in the first equation, so we can substitute it (l) into the second equation: $(2w + 5) \cdot w = 60$.
3. We can simplify this equation thus: $2w^2 + 5w - 60 = 0$.
4. Solving this quadratic equation, we get: $(w - 5)(2w + 12)$, or $w = 5$ and $w = -6$.
5. If we remember the context, we can only have one value of w. Furthermore, since this is a geometric problem, it can not be negative. Thus, our value of $w = 5$.
6. Now, we substitute this into any of our original two equations to get the value of l. In this case, I'll substitute it into the second: $l \cdot 5 = 60$ and $l = 60/5 = 15$.
7. So, of our unknowns: $l = 15$, and $w = 5$.

Step 4:

1. **Is the solution correct**? We can verify it by substituting our values for l and w into our conditions ($l = 2w + 5$ and $l \cdot w = 60$), and we will find that they are true.
2. **Could we have solved this problem (correctly) with a different method**? Is there a more concise way to explain our solution? An obvious answer to our first question is that we could have substituted the second equation into the first instead of the opposite operation. Furthermore, we could have done this in terms of w instead of l.
3. What if we knew the length, and the relation of the length to the width, instead of the area?