Notes on 'Linear Algebra, by Serge Lang'

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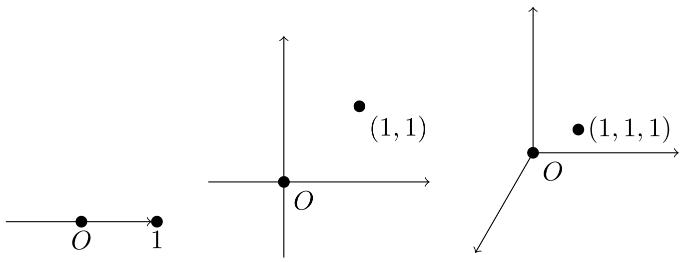
1 Vectors

1.1 Definition of Points in Space

Notes

Definition 1.1.1. A **co-ordinate** represents a point on a **line**, which we can also call 1-space. A pair of co-ordinates represent a point on a **plane**, or 2-space. A trio of co-ordinates represent a point on 3-space, and so on, such that a point with n co-ordinates (what we call an n-tuple) is a point on n**-space**. An n-tuple can be represented as $(x_1, x_2, x_3, ..., x_n)$, and we denote an n-space as $\mathbb{R}^n$.

Example 1.1.1. A point on 1-space, 2-space, and 3-space respectively.



Definition 1.1.2. We can say that higher dimensional spaces are **products** of lower dimensional spaces, the result of putting them side-by-side.

Example 1.1.2. $\mathbb{R}^3 = \mathbb{R}^2 \cdot \mathbb{R}^1$

Definition 1.1.3. To add two points, we sum each corresponding co-ordinate in each, and expresss the results as a point. For instance, if A and B are two points in n-space, their sum would be: $A + B = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$. In addition operations involving points, the following properties are observed:

1. **Associativity:** $(A + B) + C = A + (B + C)$
2. **Commutativity:** $A + B = B + A$
3. **Identity Element:** If O is the origin point, $A + O = A$
4. **Inverse Element:** Let $A = (a_1, ..., a_n)$ and $-A = (-a_1, ..., -a_n)$. Then $A+(-A) = O$

Example 1.1.3. Let the points $A = (1, 3, 5)$, $B = (-3, 4, 9)$, $C = (4, 5, 9)$, and $O = (0, 0, 0)$. The previously defined properties can be demonstrated:

1. **Associativity:**

$$\begin{aligned}(A + B) + C &= (-2, 7, 14) + (4, 5, 9) \\ &= (1, 3, 5) + (1, 9, 18) \\ &= (2, 12, 23)\end{aligned}$$

2. **Commutativity:**

$$\begin{aligned}B + A &= (-3 + 1, 4 + 3, 9 + 5) \\ &= (-2, 7, 14) \\ &= A + B\end{aligned}$$

3. **Identity Element:**

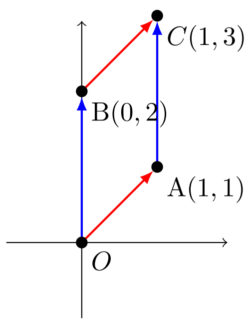
$$\begin{aligned} A + O &= (1,3,5) + (0,0,0) \\ &= (1,3,5) \\ &= A \end{aligned}$$

4. **Inverse Element:**

$$\begin{aligned} A + (-A) &= (1-1, 3-3, 5-5) \\ &= (0,0,0) \\ &= O \end{aligned}$$

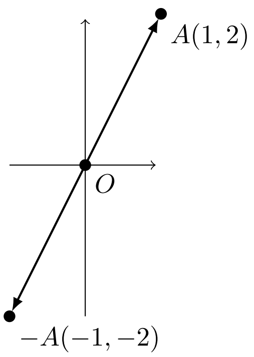
Definition 1.1.4. Representing the addition of vectors geometrically will result in a parallelogram. Vector addition leads to a parallelogram because it's like taking two different paths to reach the **same point**, and the resulting shape formed by these paths is a parallelogram.

Example 1.1.4. Let $A = (1,1)$, $B = (0,2)$, $A + B = C = (1,3)$, and $O = (0,0)$.



Definition 1.1.5. The negative of a point A, $-A$, has the exact same magnitude but an opposite direction. We call $-A$ a **reflection** of A *through* the origin.

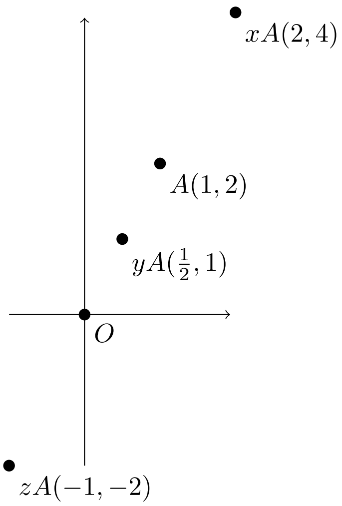
Example 1.1.5. Let $A = (1,2)$, $O = (0,0)$, and $-A = (-1,-2)$.



Definition 1.1.6. Multiplying a vector A by a scalar c gives us $cA = (ca_1, ca_2, ..., ca_n)$. Scalar multiplication of vectors follows the regular properties of multiplication: namely that $c(A + B) = cA + cB$ or $(c_1 + c_2)A = c_1 A + c_2 A$.

Geometrically, scalar multiplication of A by a scalar c **stretches** A if $c > 1$, **shrinks** A if $0 < c < 1$, and reverses its direction as well if $c < 0$.

Example 1.1.6. Let $A = (1,2)$, $x = 2$, $y = \frac{1}{2}$, and $z = -1$.



Exercises

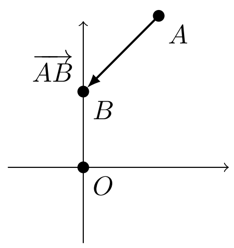
1. $A = (2, -1)$, $B = (-1, 1)$
   (a) $A + B = (1, 0)$
   (b) $A - B = (3, -2)$
   (c) $3A = (6, -3)$
   (d) $-2B = (2, -2)$
2. $A = (-1, 3)$, $B = (0, 4)$
   (a) $A + B = (-1, 7)$
   (b) $A - B = (-1, -1)$
   (c) $3A = (-3, 9)$
   (d) $-2B = (0, -8)$
3. $A = (2, -1, 5)$, $B = (-1, 1, 1)$
   (a) $A + B = (1, 0, 6)$
   (b) $A - B = (3, -2, 4)$
   (c) $3A = (6, -3, 15)$
   (d) $-2B = (2, -2, -2)$
4. $A = (-1, -2, 3)$, $B = (-1, 3, -4)$
   (a) $A + B = (-2, 1, -1)$
   (b) $A - B = (0, -5, 7)$
   (c) $3A = (-3, -6, 9)$
   (d) $-2B = (2, -6, 8)$
5. $A = (\pi, 3, -1)$, $B = (2\pi, -3, 7)$
   (a) $A + B = (3\pi, 0, 6)$
   (b) $A - B = (-\pi, 6, -8)$
   (c) $3A = (3\pi, 9, -3)$
   (d) $-2B = (-4\pi, 6, -14)$

1.2 Located Vectors

Notes

Definition 1.2.1. A **located vector** is an *ordered pair* of points which we can express as a line or an arrow. For instance, in the located vector $\overrightarrow{AB}$, A is the beginning point and B is the end point. We say that vectors with a beginning point of O are *located at the origin*; those with a beginning point of, for instance, A, are *located at* A, and so forth. $\overrightarrow{AB}$ is located at A, in this instance.

Example 1.2.1. Let $A = (1, 2)$ and $B = (0, 1)$. Thus, $\overrightarrow{AB}$ could be represented:



Definition 1.2.2. In this plane, $b_1 = a_1 + (b_1 - a_1)$ and $b_2 = a_2 + (b_2 - a_2)$. You can further rearrange these equations to get a_1 and a_2. And, from this, we can generalize that $B = A + (B - A)$.

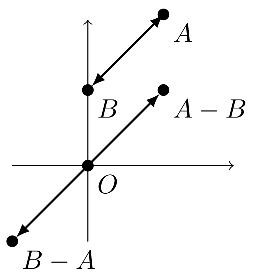
Example 1.2.2. Let $A = (1, 2)$ and $B = (0, 1)$. We can prove the previous assertions thus:

1. That $b_1 = a_1 + (b_1 - a_1)$: $0 = 1 + (0 - 1)$;
2. That $b_2 = a_2 + (b_2 - a_2)$: $1 = 2 + (1 - 2)$;
3. That $B = A + (B - A)$:

$$\begin{aligned} &= (1, 2) + ((0, 1) - (1, 2)) \\ &= (1, 2) + (-1, -1) \\ &= (0, 1) = B \end{aligned}$$

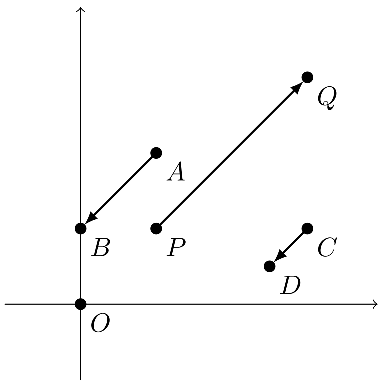
Definition 1.2.3. We say that two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are **equivalent** if $B - A = D - C$. Furthermore, *every* located vector $\overrightarrow{AB}$ can be said to have an equivalent involving the origin point O, because $\overrightarrow{AB}$ can be expressed as equivalent to $\overrightarrow{O(B - A)}$.

Example 1.2.3. Let $A = (1, 2)$, $B = (0, 1)$, and $B - A = (-1, -1)$. (Note the **direction**)



Definition 1.2.4. We say that two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are **parallel** if $B - A = c(D - C)$, where $c \neq 0$. Furthermore, if $c > 0$, then the two vectors are parallel in the **same direction**. If $c < 0$, they are parallel but in the **opposite direction**.

Example 1.2.4. Let $A = (1, 2)$, $B = (0, 1)$, $C = (3, 1)$, $D = (2.5, 0.5)$, $P = (1, 1)$, and $Q = (2, 4)$. Thus, $B - A = \frac{1}{2}(D - C)$, and the two vectors are parallel in the same direction, while $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are parallel in the opposite directions.



Exercises

1. $(3, 4) = (6, -3)$ — **Not Equivalent**
2. $(-4, 1) = (-4, 1)$ — **Equivalent**
3. $(-3, 4, -9) = (-3, 4, 9)$ — **Equivalent**
4. $(-3, 0, 9) = (-3, 0, 9)$ — **Equivalent**
5. $(3, 4) = k(8, -4)$ — **Not Parallel**
6. $(-4, 1) = -1(4, -1)$ — **Parallel**
7. $(-3, 4, -9) = \frac{1}{2}(-6, 8, -18)$ — **Parallel**
8. $(-3, 0, 9) = (-9, 0, -27)$ — **Not Parallel**

1.3 Scalar Product

Notes

Definition 1.3.1. The **scalar product**, sometimes called **dot product**, of two vectors A and B is obtained through multiplication. In 2-space, this means: $A \cdot B = a_1b_1 + a_2b_2$. In 3-space: $A \cdot B = a_1b_1 + a_2b_2 + a_3b_3$. The scalar of two vectors in n-space is thus:

$$A \cdot B = a_1b_1 + a_2b_2 + ... + a_nb_n$$

The product of such operations is always a single **number**.

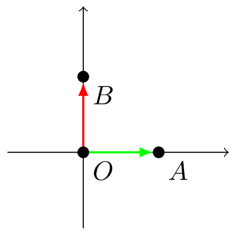
Example 1.3.1. Let $A = (1, 2)$ and $B = (0, 1)$. The scalar, or dot, product of the two vectors is: $A \cdot B = 1 \cdot 0 + 2 \cdot 1 = 2$.

Definition 1.3.2. There are four properties associated with scalar multiplication:

1. $(A + B) \cdot C = CA + CB$
2. $A \cdot B = B \cdot A$
3. If $A = O$, then $A \cdot A = 0$. Otherwise, $A \cdot A > 0$.
4. If x is a number, then: $xA \cdot B = x(A \cdot B)$.

Definition 1.3.3. Two vectors A and B are said to be **perpendicular**, or **orthogonal**, if their dot product $A \cdot B$ comes out to 0.

Example 1.3.2. Let $A = (1, 0)$ and $B = (0, 1)$. The scalar product of the two, $A \cdot B$, is thus: $A \cdot B = 1 \cdot 0 + 1 \cdot 0 = 0$. Since the result is 0, the two are vectors are said to be orthogonal, and we can express this geometrically:



Exercises

1. Find $A \cdot A$ and $A \cdot B$ for:
   (a) $A = (2, -1)$,
   $B = (-1, 1)$,
   $A \cdot A = 4 + 1 = 5$,
   $A \cdot B = 2 \cdot -1 + -1 \cdot 1 = -3$.
   (b) $A = (-1, 3)$,
   $B = (0, 4)$,
   $A \cdot A = -1 \cdot -1 + 3 \cdot 3 = 10$,
   $A \cdot B = -1 \cdot 0 + 3 \cdot 4 = 12$.
   (c) $A = (2, -1, 5)$,
   $B = (-1, 1, 1)$,
   $A \cdot A = 4 + 1 + 25 = 30$,
   $A \cdot B = 2 \cdot -1 + -1 \cdot 1 + 5 \cdot 1 = 2$.
   (d) $A = (-1, -2, 3)$,
   $B = (-1, 3, 4)$,
   $A \cdot A = -1 \cdot -1 + -2 \cdot -2 + 3 \cdot 3 = 14$,
   $A \cdot B = -1 \cdot -1 + -2 \cdot 3 + 3 \cdot 4 = 7$.

1.4 The Norm of a Vector